RESEARCH INTO ADVANCED CONCEPTS OF MICROWAVE POWER AMPLIFICATION AND GENERATION UTILIZING LINEAR BEAM DEVICES

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ABSTRACT

This is an interim report which summarizes the initial work on a theoretical study of some aspects of the interaction between a drifting stream of electrons with transverse cyclotron motions and an electromagnetic field. Particular emphasis is given to the possible generation and amplification of millimeter waves. The report includes brief discussions of the classes of interaction of interest, the structure of d-c, axially symmetric, magnetically focused, relativistic electron beams, and consideration of the electron beam as a spatially dispersive dielectric medium.

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I. INTRODUCTION

The objective of this research program is to explore theoretically some aspects of the interaction between a drifting stream of electrons having transverse cyclotron motions and an electromagnetic field; particular emphasis being given to the possible generation and amplification of millimeter waves. Because of the interest in the possible applications to millimeter wavelengths, the study concentrates on electron stream-electromagnetic field interactions which do not involve an r-f slow wave circuit structure.

This interim report summarizes the initial work on the program. It includes brief discussions of the classes of interaction of interest, the structure of d-c, axially symmetric, magnetically focused, relativistic electron beams, and consideration of the electron beam as a spatially dispersive dielectric medium.

II. CLASSES OF INTERACTION

The general classification of interactions involving a drifting electron stream with transverse cyclotron motions is very broad, even when one excludes interactions with electromagnetic waves propagating along slow wave structures (e.g., transverse wave traveling wave tubes and backward wave oscillators, magnetrons, etc.). Many of the possible devices have been included in a representative list recently enumerated and discussed relative to millimeter wave generation by Kulke and Veronda*; therefore, a listing will not be repeated here. They classify electron beam devices into three major groups: (1) periodic circuit devices, (2) periodic beam devices, and (3) beam harmonic couplers. This study is concerned with a subclass of the first class (d-c pumped transverse wave parametric amplifiers) and the class of periodic beam devices.

Among the devices or interaction schemes to be considered there are many similarities. For example, in almost all cases of amplifiers, an input coupling region, a gain region, and an output coupling region can be identified, although in some cases these regions may overlap to some extent. Also, in almost all cases

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the mechanism of gain can be qualitatively explained (using a wave picture) as the interaction between two waves. In the d-c pumped transverse wave parametric amplifier these two waves are both beam waves; i.e., two cyclotron waves, two synchronous waves, or a cyclotron and a synchronous wave. In a rippled beam amplifier, Ubitron, cyclotron resonance amplifier, etc., one wave is a beam wave and the other is a wave associated with a uniform r-f circuit or waveguide.

The most effective means of comparing these different devices in a general manner is probably on the basis of a wave analysis, although this must certainly oversimplify many aspects of the detailed interaction mechanisms. Of the several possible types of wave analysis including field solution of the boundary value problem, expansion in normal modes, and coupled mode analysis, the latter is the most suitable for a general, qualitative comparison of the various devices. The others are too detailed and unwieldy for a survey.

One of the major attractions of the d-c pumped parametric amplifier class of interaction also lies at the heart of its most serious disadvantage in practice. The major attraction is that the gain mechanism involves the interaction between two beam waves, thus in the gain region no r-f structure is required. But put in other terms, this means that the beam in the absence of an input r-f signal must be in a state of unstable equilibrium in this gain region as far as transverse displacements are concerned. When an r-f input signal perturbs the beam, this perturbation grows and the signal is amplified. The function of the d-c pump structure is to provide the d-c electric and/or magnetic fields in the gain region which cause the unstable equilibrium of the d-c beam. A major problem is to ensure that the pump fields provide instability (i.e., gain) for only the desired r-f signal perturbations and not for other d-c or a-c perturbations which are unavoidably present due to tolerance limitations, noise, stray fields, etc. And, perhaps even more difficult, to avoid instabilities for these other extraneous perturbations not only for the d-c beam, but also when the beam has been perturbed by the desired r-f input signal. This is the problem of "beam blow up" in d-c pumped transverse wave parametric amplifiers. Although various theoretical analyses have indicated that these problems may be minimized in certain ways, experimental evidence of a successful solution to these problems has not yet been obtained.

Although the d-c pumped parametric amplifier does not require an r-f circuit in the gain region, it does require a d-c pump structure which must provide a periodically varying electric or magnetic field. This is certainly a disadvantage for millimeter wavelength operation. Although the problems of r-f losses at millimeter wavelengths are avoided, the problems of machining and tolerances in the construction of periodic circuits with short periods must be solved.

It would appear then that d-c pumped parametric amplifiers have two major inherent disadvantages compared to periodic beam devices using a uniform r-f circuit in the gain region. It should be noted, however, that many of the periodic beam devices also use a periodic perturbation of the beam to produce the rippling, undulation, etc., desired. In these cases, the periodic beam devices would also have some of the same problems with a short period d-c structure that the d-c pumped parametric amplifier might have. However, the serious problem of beam blow up is not present, at least if the beam focusing and periodic perturbation are designed to avoid any parametric amplification effects in the gain region. And, of course, it is possible to produce a ripple on an electron beam by the proper control of the focusing conditions without the presence of a periodic structure, and this approach might be used.

At this time, the periodic beam class of interaction would appear to offer superior promise, in principle, for the generation and amplification of short microwaves in the millimeter wavelength region.

III. RELATIVISTIC ELECTRON BEAMS

In any study of electron stream interaction involving transverse displacements and velocities, the unperturbed d-c electron beam is important as the basic starting point for the analysis. This study is concerned with transverse interactions in beams focussed, among other methods, by a finite axial magnetic field. As a background for the interaction analysis, the focusing of an axially symmetric, uniform electron stream was briefly examined for several possible initial conditions. Because of the possible importance of relativistic effects, as for example in the electron cyclotron maser,[1] the influence of special relativity was included in this analysis. Some of the results are noted here.

A. Basic Equations

The basic equations of motion for d-c, axially symmetric electron beam can be written in cylindrical coordinates as

$$\frac{d}{dt}\left[\frac{\dot{r}}{\sqrt{1-\beta^2}}\right] = \frac{r\dot{\theta}^2}{\sqrt{1-\beta^2}} - \frac{e}{m}\left[E_r + r\dot{\theta}B_z - \dot{z}B_\theta\right], \quad \text{(1a)}$$

$$\frac{d}{dt}\left[\frac{r^2\dot{\theta}}{\sqrt{1-\beta^2}} - \frac{e}{m}\int_0^r \xi B_z(\xi)\,d\xi\right] = 0, \quad \text{(1b)}$$

$$\frac{d}{dt}\left[\frac{\dot{z}}{\sqrt{1-\beta^2}}\right] = -\frac{e}{m}\left[E_z + \dot{r}B_\theta - r\dot{\theta}B_r\right], \tag{1c}$$

$$\frac{d}{dt}\left[\frac{1}{\sqrt{1-\beta^2}} - \frac{e}{mc^2}\phi\right] = 0 \quad . \tag{1d}$$

Here e is the magnitude of the electron charge, m is the electron rest mass,

$$\beta^2 = \frac{\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2}{c^2} \tag{2}$$

ϕ is the scalar potential, and $\overline{E}$ and $\overline{B}$ are the total electric field and magnetic flux density (including both the applied and self fields).

In this case of an axially symmetric beam, it is seen from (1b) and (1d) that two quantities are conserved along the beam:

$$\frac{r^2\dot{\theta}}{\sqrt{1-\beta^2}} - \frac{e}{m}\int_0^r \xi\, B_z(\xi)\, d\xi \ , \tag{3a}$$

$$\frac{1}{\sqrt{1-\beta^2}} - \frac{e}{mc^2}\phi \quad . \tag{3b}$$

If it is assumed that the electron velocity at the cathode surface is zero, then examination of Equations (1a)-(1d) shows that this is a necessary and sufficient condition for the cathode to be a unipotential surface. This is assumed hereafter, and the cathode potential is taken to be zero. Then using the conserved quantities of (3a) and (3b), one can write

$$\frac{r^2\dot{\theta}}{\sqrt{1-\beta^2}} = \frac{e}{m}\int_0^{r} \xi B_z(\xi)d\xi - \frac{e}{m}\int_0^{r_c} \xi B_{cz}(\xi)\, d\xi, \qquad (4a)$$

$$\frac{1}{\sqrt{1-\beta^2}} = 1 + \frac{e}{mc^2}\,\phi \quad . \qquad (4b)$$

Here r_c is the radial position at the cathode ($z = 0$) of an electron which has the radial position r at an axial position z, and B_{cz} is the axial magnetic flux density at the cathode. Equation (4a) is Busch's theorem in relativistic form.

We are mainly interested in electron beams in drift regions where the applied magnetic field is only in the z direction, $B_r = 0$, $\dot{r} = 0$, and all quantities are independent of z as well as θ. The basic equations are now (1a), (4a), and (4b), since (1c) becomes identially zero. Introducing the plasma frequency

$$\omega_p = \sqrt{\frac{-\rho e}{\epsilon_o m}} \quad , \tag{5a}$$

and the cyclotron frequency associated with the applied d-c axial magnetic flux density B_{za},

$$\omega_a = \frac{e}{m} \quad B_{za} \quad , \tag{5b}$$

one can write

$$\frac{e}{m} B_\theta = - \frac{1}{c^2 r} \int_o^r \xi \dot{z}(\xi) \; \omega_p^2(\xi) \; d\xi \; , \tag{6a}$$

$$\frac{e}{m} B_z = - \frac{1}{c^2} \int_r^b \xi \dot{\theta}(\xi) \; \omega_p^2(\xi) \; d\xi + \omega_a \quad , \tag{6b}$$

where b is the beam radius. The basic equations are now

$$\frac{r\dot{\theta}^2}{\sqrt{1-\beta^2}} = - \frac{1}{r} \int_o^r \xi\omega_p^2(\xi) \; d\xi + \frac{\dot{z}}{c^2 r} \int_o^r \xi\dot{z}(\xi) \; \omega_p^2(\xi) \; d\xi$$

$$- \frac{r\dot{\theta}}{c^2} \int_r^b \xi\dot{\theta}(\xi) \; \omega_p^2(\xi) d\xi \quad + r\dot{\theta} \; \omega_a(r) \quad , \tag{7a}$$

$$\frac{r^2\dot{\theta}}{\sqrt{1-\beta^2}} = -\frac{1}{c^2}\int_0^r \xi\left[\int_\xi^b \rho\dot{\theta}(\rho)\omega_p^2(\rho)d\rho\right]d\xi + \int_0^r \xi\omega_a(\xi)\, d\xi$$

$$- \frac{e}{m}\int_0^{r_c} \xi B_{cz}(\xi)\, d\xi \quad , \tag{7b}$$

$$\frac{1}{\sqrt{1-\beta^2}} = 1 + \frac{e}{mc^2}\ \phi(r) \quad . \tag{7c}$$

B. Brillouin Flow*

The conditions under which $\dot{\theta}$ and $\dot{z}$ can be independent of r are discussed briefly. First, no magnetic flux can link the cathode, or $B_{cz} = 0$. This is the usual condition familiar from nonrelativistic theory. Note however, that B_{cz} includes both the applied axial magnetic flux density and that due to any azimuthal beam current in the vicinity of the cathode. Thus, one might have to apply some small axial magnetic field in the proper direction at the cathode to establish Brillouin flow.

The second condition is that the radial variation of the space charge density and the applied axial magnetic flux density in the drift region must vary with r such that

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*Several aspects of relativistic Brillouin flow have also been treated by DePackh and Ulrich.[2]

$$\omega_p(r) = \frac{\sqrt{2}\ \dot{\theta}}{(1-\beta^2)^{3/4}} \sqrt{1 + \frac{3}{2} \frac{\frac{r^2\dot{\theta}^2}{c^2}}{1-\beta^2}}\ , \tag{8a}$$

$$\omega_a(r) = 2\dot{\theta} \left[\frac{1}{\sqrt{1-\beta^2}} + \frac{\frac{b^2\dot{\theta}^2}{2c^2}}{(1-\beta^2(b))^{3/2}} \right]. \tag{8b}$$

Note that as the beam voltage is raised to operate further in the relativistic regime, the more the charge density is concentrated at the beam edge.

The total axial beam current is given by

$$I_o = \frac{2\pi m \epsilon_o}{e}\ \dot{z} \int_o^b \omega_p^2\,(r)\ r\ d\ r\ . \tag{9}$$

Inserting $\omega_p(r)$ from (8a), Equation (9) can be integrated to give I_o in terms of b, θ, and $\phi(b)$. One finds that for fixed b and $\phi(b)$ there is a maximum possible beam current given by

$$I_m = \frac{4}{3\sqrt{3}}\ \frac{\pi m \epsilon_o c^3}{e} \left[1 - \frac{1}{\left(1 + \frac{e}{mc^2}\,\phi(b)\right)^2} \right]^{3/2}$$

$$\left[1 + \frac{e}{mc^2}\,\phi(b) \right]^3 . \tag{10}$$

At low voltages this reduces to the familiar value $25.4 \times 10^{-6} \left|\phi(b)\right|^{3/2}$ amperes, while at very high voltages the maximum current tends to become proportional to $\left[\phi(b)\right]^3$. Figure 1 shows I_m versus $\phi(b)$ for a wide range of beam voltages.

The axial and angular velocities, suitably normalized, are plotted in Figure 2 versus the beam voltage for the maximum beam current I_m, and for a beam current of one-tenth the maximum value. The sensitivity of the axial velocity to the beam current at a given beam voltage is clearly evident. At low beam voltages, $\dot{z}/c$ and $b\dot{\theta}/c$ are proportional to $\phi^{1/2}$. At beam voltages above about 10^4 volts, $\dot{z}/c$ and $b\dot{\theta}/c$ depart from this dependence and approach assympotically to 0.577 and 0.817 respectively as ϕ approaches infinity (for maximum current).

Figure 3 shows the dependence of the beam plasma frequency and the cyclotron frequency of the applied axial magnetic field on the beam voltage for beam currents equal to the maximum value and one-tenth of the maximum value. Again, at low voltages $\omega_p b/c$ and $\omega_a b/c$ are proportional to $\phi^{1/2}$. Above about 10^4 volts, they start to increase much more rapidly with ϕ; their assymptotic dependence at large beam voltage is as $\phi^{5/2}$ for $\omega_p b/c$, and as ϕ^3 for $\omega_a b/c$.

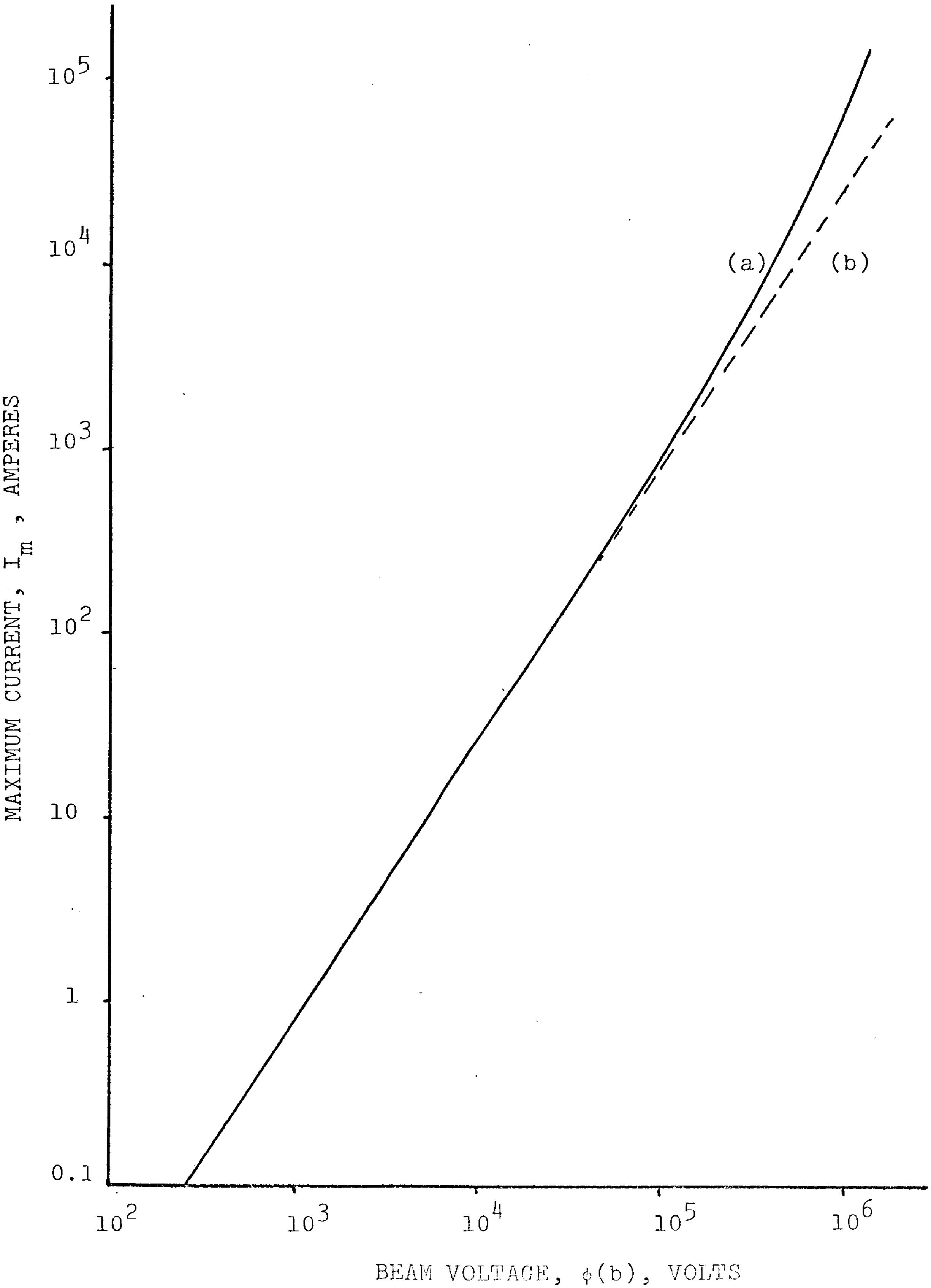

FIGURE 1. Maximum Beam Current Versus Beam Voltage for a Brillouin Beam. (a) Maximum Beam Current. (b) $25.4 \times 10^{-6}\ \phi^{3/2}$.

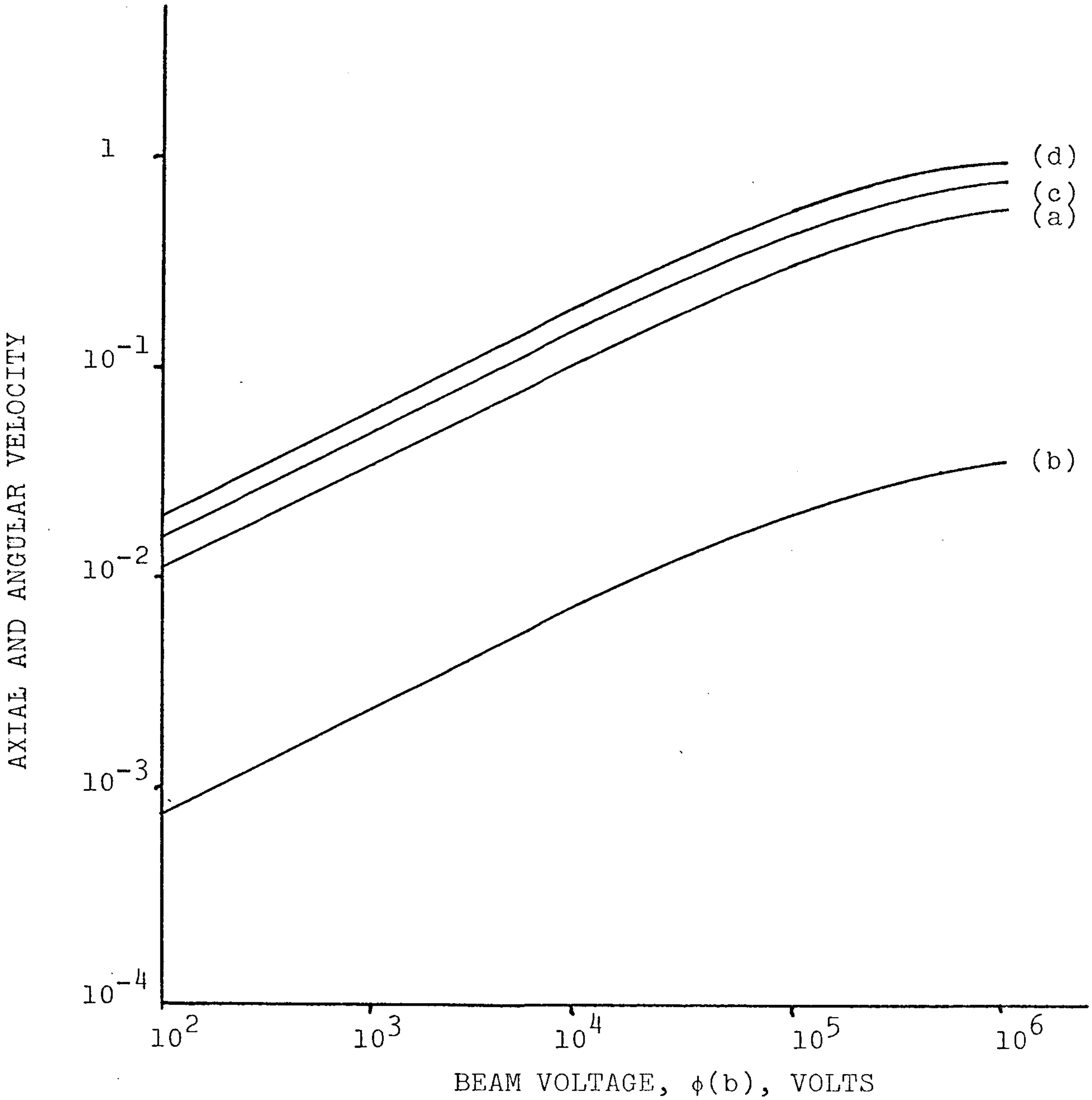

FIGURE 2. Normalized Axial and Angular Velocities Versus Beam Voltage for a Brillouin Beam With Maximum Beam Current I_m and one-tenth Maximum Beam Current $I_{m/10}$. (a) $\dot{z}/c$ for I_m. (b) $\dot{z}/c$ for $I_{m/10}$. (c) $b\dot{\theta}/c$ for I_m. (d) $b\dot{\theta}/10$ for $I_{m/10}$.

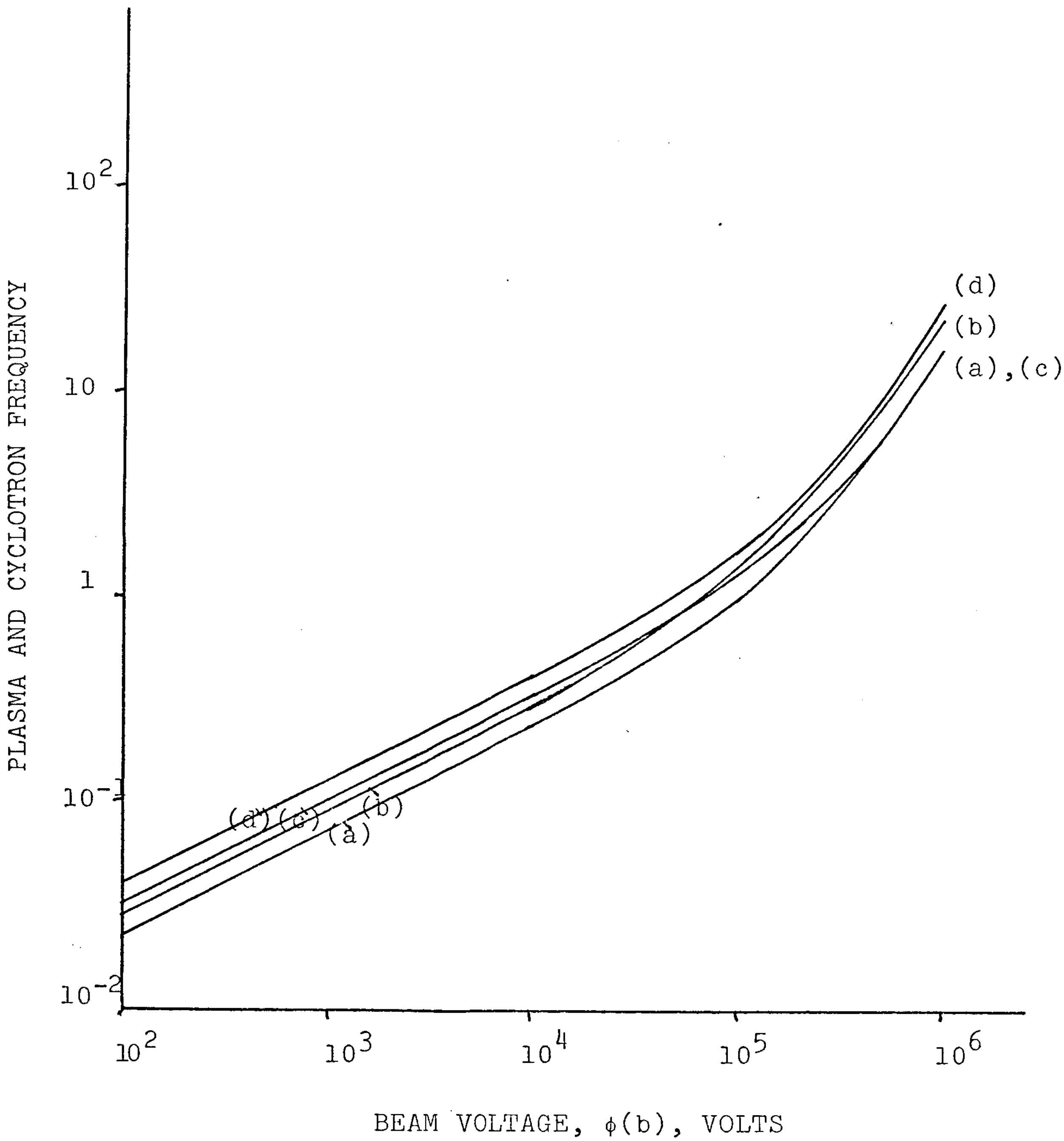

FIGURE 3. Normalized Plasma and Cyclotron Frequencies for a Brillouin Beam with Maximum Beam Current I_m and One-Tenth Maximum Beam Current $I_{m/10}$.

(a) $b\omega_{p/c}$ for I_m.

(b) $b\omega_{p/c}$ for $I_{m/10}$.

(c) $b\omega_{a/c}$ for I_m.

(d) $b\omega_{a/c}$ for $I_{m/10}$.

C. Uniform Charge Density Beam

There are, of course, an infinite variety of axially symmetric electron beams one might consider. Attention will be confined here to one class which seems of considerable practical interest. This is an electron beam in which the charge density is uniform with radius; i.e., ω_p is a constant in the drift region. Further, it is assumed that the relative axial magnetic flux density pattern is similar in both the drift region and the cathode region, so that

$$\eta = 1 - \frac{B_{cz}}{B_z} \frac{r_c^2}{r^2} \tag{11}$$

is a constant for a given beam. This parameter is defined so that $\eta = 0$ corresponds to the full magnetic flux linking the cathode, $0 < \eta < 1$ to partial magnetic flux linking the cathode, $\eta = 1$ to no magnetic flux linking the cathode, and $1 < \eta < 2$ to reversed magnetic flux linking the cathode. Finally, it is assumed for simiplicity that the total axial magnetic flux density in the drift region is independent of r. Thus $\omega_a = eB_{za}/m$ must vary across the beam in order that $\omega_c = eB_z/m$ be constant; ω_c is the cyclotron frequency including both the applied and the self magnetic fields.

With these assumptions, one finds that $\dot{\theta}$ and $\dot{z}$ must now vary with r in the drift region. For convenience, one introduces the parameter

$$\Omega = \frac{1}{\sqrt{1 - \frac{\dot{z}^2(0)}{c^2}}} \quad . \tag{12a}$$

Examination of the basic equations leads oneto conclude that this parameter can also be written as

$$\Omega = 1 + \frac{e}{mc^2}\,\phi(b) - \frac{\omega_p^2 b^2}{4c^2} \quad , \tag{12b}$$

and

$$\Omega = \frac{\omega_p^2}{\omega_c^2\left(\eta - \frac{\eta^2}{2}\right)} \quad . \tag{12c}$$

Combining Equations (7a), (7b), and (7c), and using the parameters introduced, one finds that

$$\frac{\dot{z}}{c} = \frac{\Omega^2 - 1 + \Omega\left(\frac{\omega_p^2 r}{2c}\right)^2}{\Omega\left[\Omega + \left(\frac{\omega_p r}{2c}\right)^2\right]\left\{1 - \frac{2}{\Omega\left(\frac{\omega_p r}{2c}\right)^4}\left[\left(\frac{\omega_p r}{2c}\right)^2 - \Omega \ln\left(1 + \frac{1}{\Omega}\ln\left(1 + \frac{1}{\Omega}\left(\frac{\omega_p r}{2c}\right)^2\right)\right)\right]\right\}^{1/2}} \tag{13}$$

$$\frac{\dot{\theta}}{\eta\omega_c} = \frac{\sqrt{1 - \frac{\dot{z}^2}{c^2}}}{4\sqrt{1 + \frac{\eta^2\omega_c^2}{\omega_p^2}\left(\frac{\omega_p r}{2c}\right)^2}} \tag{14}$$

The expression for the axial velocity, $\dot{z}/c$, can be simplified somewhat if it is assumed that $(\omega_p b/2c) << 1$; a condition which is not inconsistent with relativistic flow. With this approximation,

$$\frac{\dot{z}}{c} = \frac{\Omega^2 - 1 + \Omega \left(\frac{\omega_p r}{2c}\right)^2}{\left[\Omega + \left(\frac{\omega_p r}{2c}\right)^2\right]\left\{\Omega^2 - 1 + \frac{2}{3\Omega}\left(\frac{\omega_p r}{2c}\right)^2 - \frac{1}{2\Omega^2}\left(\frac{\omega_p r}{2c}\right)^4\right\}^{1/2}} . \quad (15)$$

Using this simplified form for the axial velocity, the total beam current is

$$I_o = \frac{4\pi m\epsilon_o c^3}{e}\left\{\sqrt{2}\,\Omega^2 \sin^{-1}\left[\frac{3 + \frac{3e}{mc^2}\phi(b) - 5\Omega}{\Omega\sqrt{18\,\Omega^2 - 14}}\right] + \sqrt{2}\,\Omega^2 \sin^{-1}\left[\frac{2}{\sqrt{18\,\Omega^2 - 14}}\right]\right.$$

$$- \sqrt{\frac{6}{13 - 6\,\Omega^2}}\,\sin^{-1}\left[\frac{5 + \frac{5e}{mc^2}\phi(b) + 6\Omega^3 - 13\Omega}{\left(1 + \frac{e}{mc^2}\phi(b)\right)\sqrt{18\,\Omega^2 - 14}}\right]$$

$$\left. + \sqrt{\frac{6}{13 - 6\Omega^2}}\,\sin^{-1}\left[\frac{6\Omega^2 - 8}{\sqrt{18\Omega^2 - 14}}\right]\right\} . \quad (16)$$

Several observations can be drawn from the expression for the total beam current, Equation (16). First, for fixed beam voltage $\phi(b)$, the maximum value of total beam current occurs for $\Omega = 1$ (note that $\Omega \geq 1$). This means that the maximum current is obtained when the axial velocity at the center of the beam is zero. In terms of current density the beam is essentially a hollow beam, although the charge density is uniform with radius. A second observation is that in the nonrelativistic limit, the maximum current is given by (still for $\Omega = 1$)

$$I_m = 57.1 \times 10^{-6} \left[\phi(b)\right]^{3/2}, \tag{17}$$

a value which is 2.25 times larger than for a Brillouin beam of the same radius and voltage. On the other hand, at very large beam voltages the current varies more slowly with the beam voltage; the assymptotic limit is that the current varies as the first power of the beam voltage.

The relationship between the angular velocity at the beam center and the plasma frequency is displayed in Figure 4 where $\dot{\theta}(0)/\omega_c$ is plotted versus ω_p/ω_c for several values of axial velocity at the beam center. Note that the point $\dot{\theta}(0)\omega_c = 1$ specifies a condition in which no magnetic flux links the cathode. This is

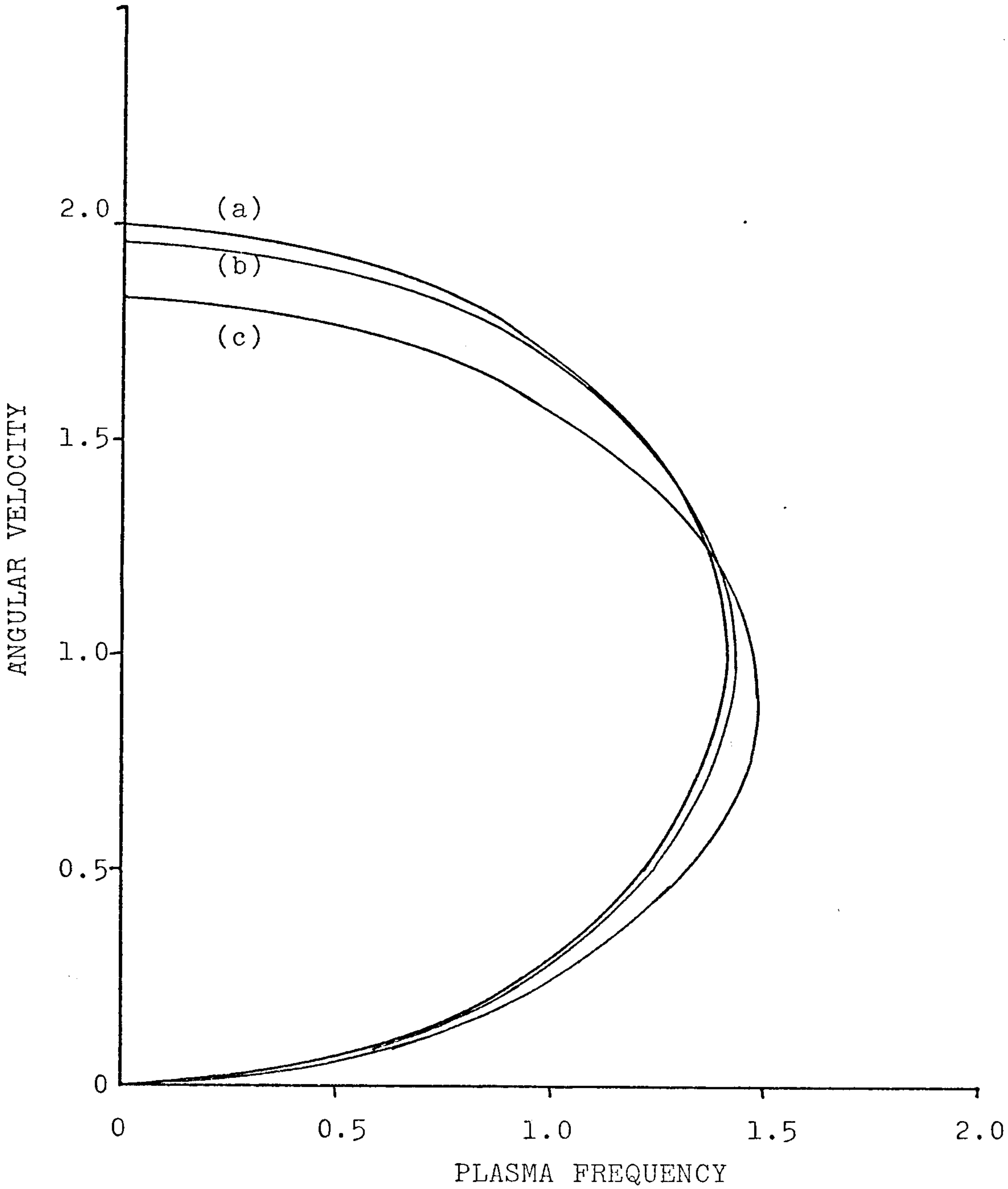

FIGURE 4. Normalized Angular Velocity $\dot{\theta}(o)/\omega_c$ Versus Normalized Plasma Frequency ω_p/ω_c For an Uniform Charge Density Beam.
(a) $\dot{z}(o)/c = 0$,
(b) $\dot{z}(o)/c = 0.197$,
(c) $\dot{z}(o)/c = 0.416$.

not referred to here as Brillouin flow; that term is reserved (perhaps arbitrarily) for the case where $\dot{z}$ and $\dot{\theta}$ are independent of r in addition to having no magnetic flux linking the cathode.

Figures 5 and 6 show the variation of the axial and angular velocities across the beam for various values of Ω and η (in the case of the angular velocity). Since the charge density is constant across the beam, the curve of $\dot{z}/c$ is proportional to the axial current density; it displays the degree of "hollowness" of the beam for various values of Ω.

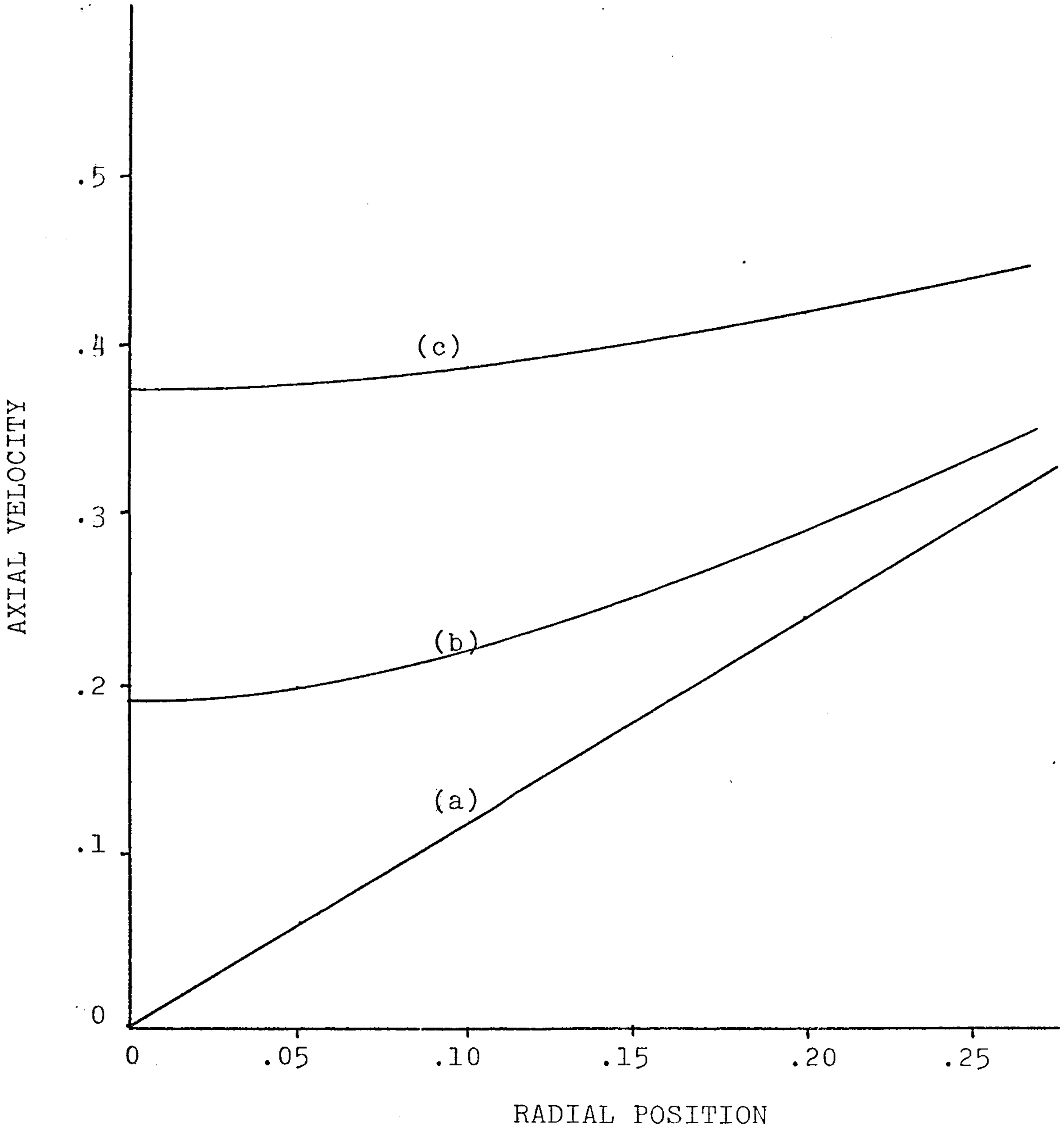

FIGURE 5. Normalized Axial Velocity $\dot{z}(r)/c$ Versus Normalized Radial Position $\omega_p r/_{2c}$ for an Uniform Charge density beam.

(a) $\Omega = 1.0$.
(b) $\Omega = 1.02$.
(c) $\Omega = 1.1$.

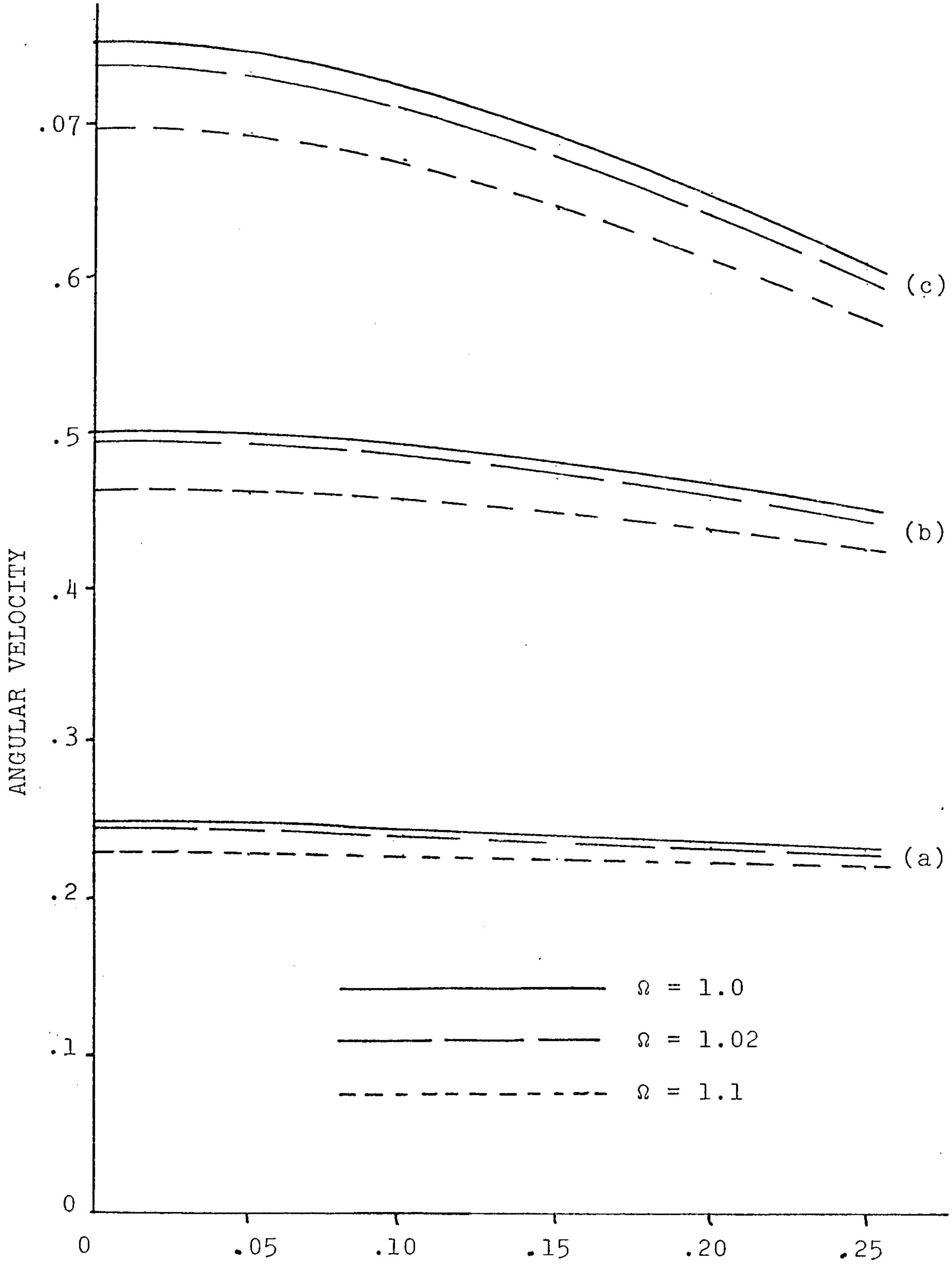

FIGURE 6. Normalized Angular Velocity $\dot{\theta}(r)/\omega_c$ Versus Normalized Radial Position $\omega_p r/_{2c}$ for an Uniform Charge Density Beam. (a) η = 0.5, Forward Magnetic Flux Through Cathode. (b) η = 1.0, No Magnetic Flux Through Cathode. (c) η = 1.5, Reversed Magnetic Flux Through Cathode.

IV. ELECTRON BEAM AS A SPATIALLY DISPERSIVE MEDIUM

It is desirable for some small signal analyses of electron beam interaction to treat the electron beam as a dielectric medium. When this procedure is used, the a-c beam velocity, current density, and charge density are removed from the interaction equations by expressing them in terms of the beam polarization $\overline{P}$. This quantity is introduced to account for the a-c perturbation of the electrons from their positions in the absence of an r-f signal. For example, when the electron motion is confined to the axial direction, the a-c beam velocity, current density, and charge density are given by

$$v = \frac{1}{\rho_o} \frac{dP}{dt} \ , \tag{18a}$$

$$J = \frac{\partial P}{\partial t} \ , \tag{18b}$$

$$P = - \frac{\partial P}{\partial z} \ . \tag{18c}$$

The introduction of the beam polarization allows the beam to be treated as a dielectric medium, and the solution of a beam-circuit interaction problem consists of the solution of Maxwell's equations with the appropriate boundary conditions. The beam current density and charge density do not appear explicitly in Maxwell's equations since they have been absorbed into the electric polarization. The equations of motion for the beam are used to establish the constitutive realtions for the medium; that is, the dependence of $\overline{P}$ on the electric field (and possibly the magnetic field) which gives the effective susceptibility. In general, the susceptibility will be a tensor.

It is found that the electron beam when considered as a dielectric medium has spatial as well as temporal dispersion. This means that the susceptibility depends explicitly on the propagation constant in addition to the frequency. There are several consequences of the dispersive character of the medium.[3] Two of the most important are that the time average stored energy density and the time average density of power flow for a medium of infinite extent with wave propagation in the z direction are given by

$$W = \frac{1}{4}\left\{ \overline{E}^* \cdot \frac{\partial(\omega\overline{\varepsilon})}{\partial\omega} \cdot \overline{E} + \overline{H}^* \cdot \frac{\partial(\omega\overline{\mu})}{\partial\omega} \cdot \overline{H} \right\}, \qquad (19a)$$

$$P = \frac{1}{4}\left\{ \overline{E} \times \overline{H}^* + \overline{E}^* \times \overline{H} - \overline{a}_z \left[\overline{E}^* \cdot \frac{\partial(\omega\overline{\varepsilon})}{\partial\beta} \cdot \overline{E} + \overline{H}^* \cdot \frac{\partial(\omega\overline{\mu})}{\partial\beta} \cdot \overline{H} \right] \right\}. \qquad (19b)$$

These generalized expressions include the stored energy and power flow associated with the sources of the dispersion in the medium. Thus in the case of an electron beam, these expressions include the kinetic energy and power flow due to the a-c motion of the electrons. Since the beam velocity and current density do not appear explicitly in this formation, all of the contributions to the energy and power flow due to the mechanical motion of the electrons must be included via the effective susceptibility of the electron beam. In effect, Equation (19b) is the small signal power theorem for the beam.

The application of this method of analysis to an electron beam of infinite cross section has been treated in some detail previously.[4,5] This discussion points out that the expressions for the time average energy density and power flow, (19a) and (19b), hold as well for an electron beam of finite cross section if the transverse boundary of the beam has an impedance independent of the frequency and propagation constant. A short circuit and an open circuit are the two most important examples. In other words, (19a) and (19b) hold if the system considered is homogeneous; for example, an electron beam filling a drift tube.

The physical reason that (19a) and (19b) are valid for the finite system considered, although they were derived for a system of infinite cross section with no transverse variation of the fields, is easily seen. In the present case, although the fields will vary in the transverse direction, because of the homogeneity of the system the transverse variation of the fields will be independent of both the frequency and the propagation constant. For example, a possible field component might vary as

$$A(\omega,\beta) \sin \left(\frac{n\pi x}{a}\right) e^{j\omega t - j\beta z}$$

or perhaps,

$$A(\omega,\beta) J_n \left(R_m \frac{r}{a}\right) \cos (n\theta) e^{j\omega t - j\beta z}$$

where R_m is a root of $J_n(R_m) = 0$. Regardless of the particular values of the frequency and propagation constant, the transverse variation remains the same.

Thus for a given mode, the transverse variation is fixed, and the parameters describing this variation (sometimes termed the transverse propagation constants) are constants. The effective susceptibility of the medium for a particular mode will depend only on the frequency and the longitudinal propagation constant since the transverse variation of the mode is fixed.

Therefore, the expressions given above for a dispersive medium of infinite cross section also apply to a mode of a homogeneous system of finite cross section.

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